THE FUZZY P-MEDIAN PROBLEM:
A MODEL TO COVER DEMANDS AND ITS SOLUTIONS

by

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ABSTRACT

We know that there are various forms of the fuzzy p-median problems so that a fuzzy version of the problem makes sense if we assume that the demands \( w_j \) to be covered are not fixed but the decision-maker has a degree of freedom in order to modify them. In this paper we consider the fuzzy p-median problem in which the demands \( w_j \) to be covered are not fixed but we want demand \( w_s \) to be covered is fixed.

KEYWORDS
Location, Linear Problem, Fuzzy Location, Fuzzy Linear Problem, Networks

INTRODUCTION

We know that the crisp p-median problem can be modeled by a mixed integer 0-1 problem [1,2]

\[
\begin{align*}
\text{Min} & \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} \\
\text{s.t.} & \\
\sum_{i=1}^{n} x_{ij} &= w_j & \forall j = 1,2,\ldots,n \\
0 \leq x_{ij} \leq w_j y_i & & 1 \leq i, j \leq n \\
\sum_{i=1}^{n} y_i &= p \\
y_i & \in \{0,1\} & & 1 \leq i \leq n
\end{align*}
\]

(1)

Where \( x_{ij} \) is the demand of vertex \( v_i \) covered by the facility at vertex \( v_j \) (if any) \( y_j \) is 1 if there is a facility at vertex \( v_j \) and 0 otherwise, \( w_j \) is the demand at vertex \( v_j \), \( d_{ij} \) is the distance from \( v_i \) to \( v_j \).

Also there are various forms of the fuzzy p-median problems [3,4,5] that a fuzzy version of the problem makes sense that the demands \( w_j \) to be covered are not fixed, and decision-maker has a degree of freedom in order to modify them. But we want demand \( w_s \) to be covered is fixed.

**Definition 1.** Assume that we given a fuzzy goal \( \widetilde{G} \) and a fuzzy constraint \( \widetilde{C} \) in a space of alternatives \( X \). Then \( \widetilde{G} \) and \( \widetilde{C} \) combine to form a decision \( \widetilde{D} \) which is a fuzzy set resulting from intersection of \( \widetilde{G} \) and \( \widetilde{C} \), i.e.,

\[ \widetilde{D} = \widetilde{G} \cap \widetilde{C} \]
And correspondingly

\[ \mu_D = \text{Min}\{ \mu_D, \mu_C \} \]

Note that the intersection of fuzzy sets is defined in the possibilistic sense by the min-operator [6].

If we want a final crisp decision, we look for a solution where \( \mu_D \) is maximum.

We define \( \mu_f \) (degree of feasibility) and \( \mu_g \) (degree of improvement of the goal), for each solution \( (x_i, y_i) \) as:

\[ \mu_f = (x_i, y_i) = h_f \left( \sum_{j=1}^{n} w_j - \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \right) \]  

(2)

That \( h_f \) is an auxiliary function given by

\[ h_f(x) = \begin{cases} 
1 & x < 0 \\
1 - \frac{x}{p_f} & 0 \leq x \leq p_f \\
0 & x > p_f 
\end{cases} \]  

(3)

That \( p_f \) represents the maximum tolerance level for a solution be considered feasible.

And also:

\[ \mu_g (x_i, y_i) = h_g \left( Z' - \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} \right) \]  

(4)

That \( Z' \) is crisp optimum cost, And \( h_g \) is another auxiliary function given by

\[ h_g(x) = \begin{cases} 
1 & x < 0 \\
\frac{x}{p_g} & 0 \leq x \leq p_g \\
0 & x > p_g 
\end{cases} \]  

(5)

That \( p_g \) indicates how much the cost should be reduced in order for the improvement to be considered as completely satisfactory.
Therefore, fuzzy $p$-median problem is as:

Find $(x_{ij}, y_i)$

s.t.

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} d_{ij} \leq Z^*
$$

$$
\sum_{i=1}^{n} x_{ij} \leq w_j \quad \forall j ; j = \{1,2,\ldots,n\} - \{s\}
$$

$$
\sum_{i=1}^{n} x_{is} = w_s
$$

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \geq \sum_{j=1}^{n} w_j
$$

$$
0 \leq x_{ij} \leq w_j y_i \quad 1 \leq i, j \leq n
$$

$$
\sum_{i=1}^{n} y_i = p
$$

$$
y_i \in \{0,1\} \quad 1 \leq i \leq n
$$

Assume that $\lambda$ is the membership degree to the decision set $\widetilde{D}$.

On the other hand, $\lambda$ is the global degree of satisfaction of a given solution. Therefore, we can find solutions with the greatest value of with the following auxiliary crisp problem:

Max $\lambda$

s.t.

$$
\lambda \leq \mu_g(x_{ij}, y_i)
$$

$$
\lambda \leq \mu_f(x_{ij}, y_i)
$$

$$
0 \leq x_{ij} \leq w_j y_i \quad 1 \leq i, j \leq n
$$

$$
\sum_{i=1}^{n} x_{ij} \leq w_j \quad \forall j ; j = \{1,2,\ldots,n\} - \{s\}
$$

$$
\sum_{i=1}^{n} x_{is} = w_s
$$

$$
\sum_{i=1}^{n} y_i = p
$$

$$
y_i \in \{0,1\} \quad 1 \leq i \leq n \quad 0 \leq \lambda \leq 1$$
That if we replace $\mu_f$ and $\mu_g$ by their definitions then above auxiliary crisp problem will as:

$$\begin{align*}
\text{Max} & \quad \lambda \\
\text{s.t.} & \quad \lambda + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_{ij}d_{ij}}{p_g} x_{ij} \leq \frac{Z^*}{p_g} \\
& \quad \lambda + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{p_f} x_{ij} \leq 1 - \frac{\sum_{j=1}^{n} w_j}{p_f} \\
0 & \leq x_{ij} \leq w_j y_i \quad 1 \leq i, j \leq n \\
\sum_{i=1}^{n} x_{ij} & \leq w_j \quad \forall j : j = \{1,2,\ldots,n\} - \{s\} \\
\sum_{i=1}^{n} x_{is} & = w_s \\
\sum_{i=1}^{n} y_i & = p \quad y_i \in \{0,1\} \quad 1 \leq i \leq n \quad 0 \leq \lambda \leq 1
\end{align*}$$

(8)

This problem can be solved by a standard branch-and-bound. When the number of vertices is not too large, we can use at enumeration algorithm [3].

Remark: We do not consider vertex vs in step 1 enumeration algorithm [3].

(i.e. We do not consider vertex $v_i$ in order all vertices according to decreasing values of $D_{ij}$ (That $D_{ij}$ is distance vertex $v_i$ to facility $v_j$).)

Example: Consider a 2-median problem in the six-vertex network given in

FIGURE 1
A SIX VERTEX NETWORK

5
\hspace{1cm} 400
\hspace{1cm} 6

\begin{align*}
5 & \quad 3 \quad 400 \\
3 & \quad 5 \\
4 & \quad 4 \\
3 & \quad 3
\end{align*}

\begin{align*}
w_1 &= 12/5 & w_2 &= 9 \\
w_3 &= 4 & w_4 &= 8 \\
w_5 &= 9 & w_6 &= 6/5
\end{align*}
The optimal crisp vertices are vertices 2 and 6 with cost 2640. Now suppose that we want to study the possibility of reducing this cost at about 100 units and also the reduction of the covered demand to be about 1 unit but we want demand \( w_2 \) to be covered is fixed, then we consider \( p_y = 200 \) and \( p_f = 2 \)

be regard to enumeration algorithm [3].

### TABLE 1

**COMPARATION OF CRISP AND FUZZY SOLUTIONS**

<table>
<thead>
<tr>
<th>Location</th>
<th>Crisp</th>
<th>Fuzzy</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1 )</td>
<td>12/5</td>
<td>12/5</td>
<td>0</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>9</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>( W_3 )</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>( W_4 )</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>( W_5 )</td>
<td>9</td>
<td>5/95</td>
<td>-5/55</td>
</tr>
<tr>
<td>( W_6 )</td>
<td>6/5</td>
<td>2495/2</td>
<td>-144/8</td>
</tr>
<tr>
<td>( Z' )</td>
<td>2640</td>
<td>2495/2</td>
<td>-0/55</td>
</tr>
<tr>
<td>Location</td>
<td>2, 6</td>
<td>1, 5</td>
<td>( \lambda = 72/4% )</td>
</tr>
</tbody>
</table>

### CONCLUSION

If we consider all demands \( w_j (j = 1, 2, \ldots, 6) \) to be covered not fixed then fuzzy p-median problem solution will not change. Also if the demands \( w_j (j = 1, 2, 3, 4, 5) \) to be covered is fixed then fuzzy p-median problem solution will not change. But if we want demand \( w_6 \) to be covered is fixed then fuzzy p-median problem solution will be worse.

### REFERENCES

S.L.Hakimi, Optimum locations of switching centers and the absolute centers and medians of a graph, Operations Research 12 (1964)


